PERGAMON

International Journal of Heat and Mass Transfer 46 (2003) 781–789

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Influence of anisotropy on convection in porous media with nonuniform thermal gradient

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Received 10 July 2001; received in revised form 28 August 2002

Abstract

Effect of anisotropy on the onset of natural convection heat transfer in a fluid saturated porous horizontal cavity subjected to nonuniform thermal gradients is investigated analytically. The porous layer is heated from the bottom by a constant heat flux while the other surfaces are being insulated. The horizontal boundaries are either rigid/rigid or stress-free/stress-free. The hydrodynamic anisotropy of the porous matrix is considered. The principal directions of the permeability are oriented in a direction that is oblique to the gravity. Based on a parallel flow assumption, closed-form solution for the flow and heat transfer variables, valid for the onset of convection corresponding to vanishingly small wave number, is obtained in terms of the Darcy–Rayleigh number Ra, the Darcy number Da, and the anisotropic parameters K^* and θ. The critical Rayleigh number for the onset of convection arising from sudden heating and cooling at the boundaries is also predicted. The limiting cases $Da \to \infty$ (for a viscous pure fluid) and $Da \to 0$ (for anisotropic porous media) completed all results. It is demonstrated that effects of anisotropic parameters are strongly significant.

1. Introduction

Natural convection in fluid-saturated porous media bounded by two-parallel plates maintaining an adverse temperature gradient is of considerable importance in engineering fields. Applications include convection in the earth's crust, flows in soils, aquifers, petroleum extraction, storage of agriculture products, insulation techniques, and geothermal energy extraction. The state of the art has been reviewed by Cheng [1].

It is well known that the subsurface ground water possesses a general upward convective drift, due to buoyancy induced by the high underground temperature or by a high heat flux. Since the rising ground water is cooled as it approaches the surface, where heat is removed by evaporation, radiation and movement in surface streams, an unstable state may be induced, and complicated convective motions appear in the layers near the surface. It is the purpose of many scientific papers which investigate the conditions for stability or instability to occur. Thus it is of considerable interest to determine just when the fluid in a finite porous region heated from below might be unstable (convecting). Most of the work on the onset of convection in a porous medium is based on linear theory. The critical Raleigh number derived by such a theory gives a necessary condition for stability (or, equivalently, a sufficient condition for instability). The early work by Lapwood [2] determined the conditions for the onset of convection in a porous medium with horizontal isothermal boundaries. On the basis of a linear stability analysis it was found that convection occurs at Rayleigh numbers above $4\pi^2$. This result has been confirmed experimentally by many investigators. The stability of convection in a horizontal porous layer, subjected to an inclined temperature gradient of finite amplitude, was investigated by Weber [3] and Nield [4] respectively. The results showed by the critical Rayleigh number are always higher than $4\pi^2$. The stability of horizontal porous and

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0017-9310/03/$ - see front matter
PII: S0017-9310(02)00352-6

Nomenclature

A	aspect ratio of the cavity, H'/L'
a, b, c	constants, Eq. (8)
C	dimensionless temperature gradient in x direction
Da	Darcy number, K_1/L'^2
$\vec{g}$	gravitational acceleration
H'	depth of cavity
k	thermal conductivity
$\overline{\overline{K}}$	flow permeability tensor, Eq. (4)
K_1, K_2	flow permeability along the principal axes
K^*	anisotropic permeability ratio, K_1/K_2
L'	width of cavity
Nu	Nusselt number, Eq. (22)
q'	uniform heat flux
Ra	Darcy–Rayleigh number, $g\beta K_1 L'^2 q'/k\alpha\nu$
Ra^*	Rayleigh number for a fluid, Ra/Da
Ra_c	critical Rayleigh number for a porous medium
Ra_c^*	critical Rayleigh number for a fluid
S	dimensionless uniform heat sink
t	dimensionless time
$\widetilde{T}$	dimensionless temperature
T	dimensionless quasi-state temperature, $\widetilde{T} - St$
$\Delta T'$	temperature scale, $q'L'/k$
ΔT	wall to wall dimensionless temperature difference at $x = 0$
$\vec{V}$	seepage velocity
u, v	dimensionless velocity components in x, y directions
x	dimensionless horizontal coordinate
y	dimensionless vertical coordinate

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient of the fluid
θ	y-dependent temperature term
γ	inclination of the principal axis
μ	dynamic viscosity of the fluid
ν	kinematic viscosity of the fluid
ϱ	density of the fluid
$(\varrho c)_f$	heat capacity of the fluid
$(\varrho c)_p$	heat capacity of saturated porous medium
ς	dimensionless Darcy parameter, $Da^{-1/2}$
σ	heat capacity ratio, $(\varrho c)_p/(\varrho c)_f$
ψ	dimensionless stream function, ψ'/α

Superscript

$'$	dimensional quantities

Subscript

o	refers to origin

viscous layer, when the thermal gradient is not uniform, has been considered by Nield [5]. Using a Galerkin method, critical Rayleigh numbers were predicted by this author for various nonlinear basic temperature distributions and constant-flux conditions at both horizontal boundaries. Walker and Homsy [6] have used the Brinkman model to determine the critical Rayleigh number against Darcy number for the case of conducting no-slip boundaries. Rudraiah et al. [7] have considered the Brinkman model to study the onset of convection with nonlinear basic temperature profiles, using a single-term Galerkin expansion and have found that the critical Rayleigh numbers obtained were in good agreement in very large Darcy numbers, with the values reported by Nield [5]. Recently, Vasseur and Robillard [8] have used the Brinkman model to investigate the effects of nonlinear temperature distribution on stability and natural convection in a horizontal porous layer heated from below. On the basis of a parallel flow assumption, the critical Rayleigh number for the onset of convection arising from sudden heating or cooling at the boundaries is predicted.

All previous studies have usually been concerned with homogeneous isotropic porous structures. But the inclusion of more physical realism in the matrix properties of the medium is important for the accurate modeling of the anisotropic media. Anisotropy, which is generally a consequence of a preferential orientation or asymmetric geometry of the grain of fibers, is in fact encountered in numerous systems in industry and nature. A few works on the topic are concerned with the study of natural convection in anisotropic porous layers heated from below. The critical Rayleigh number for the onset of convection was first considered by Castinel and Combarnous [9] who conducted an experimental and theoretical investigation for a layer with impermeable boundaries. The effect of thermal anisotropy on the onset motion was studied by Ephere [10]. Kvernvold and Tyvand [11] extended these analyses to supercritical finite-amplitude convection. McKibbin [12] conducted an extensive study on the effects of anisotropy on the convective stability of a porous layer. At the upper surface of the porous medium, this author considered boundary conditions sufficiently general to allow both impermeable and constant-pressure boundaries. Tyvand and Storesletten [13] investigated the problem concerning the onset of the convection in an anisotropic porous layer in which the principal axes were obliquely oriented to the gravity vector. As a result, new flow patterns with a titled plane of motion or titled lateral cell walls were obtained. Also, it was demonstrated that the critical Rayleigh number was always reduced when compared with a perpendicular or parallel orientation of fibers vs boundaries. The effect of anisotropy of thermal instability in a fluid-saturated porous medium subjected to an inclined temperature gradient of finite magnitude was analyzed by Parthiban and Patil [14] using the Galerkin method. It was found that anisotropic medium is most stable while either the isotropic situation or the horizontally isotropic situation is the most unstable one depending on the horizontal Rayleigh number and anisotropy parameters.

The purpose of the present study is to determine the critical Rayleigh numbers for the onset of convection, using the parallel flow analysis, on the basis of the generalized Brinkman-extended Darcy model which allows the no-slip boundary condition on a solid wall, to be satisfied. In the first part, taking account for the anisotropy in permeability of the porous matrix, a horizontal porous layer heated from below by a constant flux, the other surfaces being insulated, is considered with two hydrodynamic boundary conditions on the upper and lower surfaces. It is demonstrated that anisotropic parameters have a considerable influence on the onset of convection.

2. Mathematical formulation

The physical model considered here consists of a two-dimensional horizontal rectangular enclosure of elongated shape filled with a porous medium composed of sparse distribution of particles completely surrounded by Boussinesq fluid. As considered in nature in fact, the porous medium is anisotropic in flow permeability and assumed isotropic in thermal conductivity. The permeabilities along the two principal axes of the porous matrix are denoted by K_1 and K_2. The anisotropy in flow permeability of the porous medium is characterized by the permeability ratio $K^* = K_1/K_2$ and the orientation angle γ, defined as the angle between the horizontal direction and the principal axis with the permeability K_2. The porous rectangular cavity is bounded by two rigid vertical side walls and the two long horizontal boundaries at $y' = 0$ and L' that may be both rigid or both stress-free. The layer is heated from the bottom by a constant heat flux q', the other surfaces are being insulated. The saturating fluid is viscous, incompressible and assumed to be everywhere in local thermodynamic equilibrium with the porous medium.

Under the above approximations, the equations describing the laminar and two-dimensional convective flow in an anisotropic porous medium can be written as follows (Bear [15], Degan and Vasseur [16])

$$\nabla \cdot \vec{V}' = 0 \tag{1}$$

$$\vec{V}' = \frac{\overline{\overline{K}}}{\mu}\left(-\nabla p' + \varrho\vec{g} + \mu_{\mathrm{eff}}\nabla^2\vec{V}'\right) \tag{2}$$

$$(\varrho c)_{\mathrm{p}}\frac{\partial T'}{\partial t'} + (\varrho c)_{\mathrm{f}}\nabla\cdot(\vec{V}'T') = k\nabla^2 T' \tag{3}$$

where $\vec{V}'$ is the superficial flow velocity, T' the temperature, $\vec{g}$ the gravitational acceleration, t' the time, $(\varrho c)_{\mathrm{p}}$ and $(\varrho c)_{\mathrm{f}}$ the heat capacities of the saturated porous medium and the fluid, respectively, μ the dynamic viscosity, μ_{eff} apparent dynamic viscosity for Brinkman's model, k the thermal conductivity, ϱ the density. The symmetrical second-order permeability tensor $\overline{\overline{K}}$ is defined as

$$\overline{\overline{K}} = \begin{bmatrix} K_2\cos^2\gamma + K_1\sin^2\gamma & (K_2 - K_1)\sin\gamma\cos\gamma \\ (K_2 - K_1)\sin\gamma\cos\gamma & K_1\cos^2\gamma + K_2\sin^2\gamma \end{bmatrix} \tag{4}$$

Introducing the Boussinesq approximation

$$\varrho = \varrho_{\mathrm{o}}[1 - \beta(T' - T'_{\mathrm{o}})] \tag{5}$$

and eliminating the pressure term in the momentum equation in the usual way and taking L', α/L', $\Delta T' = q'L'/k$, ψ'/α and $\sigma L'^2/\alpha$ (where $\alpha = k/(\varrho c)_{\mathrm{f}}$ and $\sigma = (\varrho c)_{\mathrm{p}}/\varrho c)_{\mathrm{f}}$) as respective dimensional scales for length, velocity, temperature, stream function and time, the governing equations may be written in nondimensional form as

$$a\frac{\partial^2\psi}{\partial x^2} + c\frac{\partial^2\psi}{\partial x\partial y} + b\frac{\partial^2\psi}{\partial y^2} = \lambda Da\nabla^4\psi - Ra\frac{\partial\widetilde{T}}{\partial x} \tag{6}$$

$$\frac{\partial^2\widetilde{T}}{\partial x^2} + \frac{\partial^2\widetilde{T}}{\partial y^2} = \frac{\partial\widetilde{T}}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\widetilde{T}}{\partial y}\frac{\partial\psi}{\partial x} + \frac{\partial^2\widetilde{T}}{\partial t} \tag{7}$$

where

$$\begin{cases} a = \cos^2\gamma + K^*\sin^2\gamma \\ b = \sin^2\gamma + K^*\cos^2\gamma \\ c = (1 - K^*)\sin 2\gamma \end{cases} \tag{8}$$

In the above equations, $Da = K_1/L'^2$ is the Darcy number, $Ra = K_1 g\beta L'^2 q'/(k\alpha\nu)$ the Darcy–Rayleigh number based on permeability K_1, $K^* = K_1/K_2$ the permeability ratio, and $\lambda = \mu_{\mathrm{eff}}/\mu$ the relative viscosity. In the present study, $\lambda = 1$ as a first approximation in Brinkman's extension for which $\mu_{\mathrm{eff}} \approx \mu$.

The quasi-state of the resulting transient natural convection heat transfer in the present study will be reached if the heating process is maintained long enough. So, all quantities governing the phenomenon become nearly independent of time, except the temperature which continues to increase with time. Consequently, assuming that $T = \widetilde{T} - St$, governing equations become at quasi-steady-state

$$a\frac{\partial^2\psi}{\partial x^2} + c\frac{\partial^2\psi}{\partial x\partial y} + b\frac{\partial^2\psi}{\partial y^2} = Da\nabla^4\psi - Ra\frac{\partial T}{\partial x} \tag{9}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{\partial T}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial \psi}{\partial x}+S \tag{10}$$

where S is a heat sink term.

3. Analytical solution

As discussed in detail by Cormack et al. [17], Vasseur et al. [18] and other authors, an appropriate solution to the above problem can be sought in the case of a long shallow cavity (i.e., when the aspect ratio $A(=H'/L')\to\infty$). In this limit the flow in the central part of the cavity can be assumed to be parallel and in the x-direction. Consequently, the flow and the temperature fields must be of the following form:

$$\psi(x,y)=\psi(y) \tag{11}$$

and

$$T(x,y)=Cx+\theta(y) \tag{12}$$

where C is the unknown but constant temperature gradient in the x-direction. Substituting Eqs. (9) and (10), one obtains respectively

$$\frac{d^4\psi}{dy^4}-b\varsigma^2\frac{d^2\psi}{dy^2}=\varsigma^2 RaC \tag{13}$$

and

$$\frac{d^2\theta}{dy^2}=C\frac{d\psi}{dy}+S \tag{14}$$

where $\varsigma^2=1/Da$.

The solutions to the above equations are of the form

$$\psi=RaCF_b(y) \tag{15}$$

$$\theta=RaC^2G_b(y)+\theta_c \tag{16}$$

where $F_b(y)$ and $G_b(x)$ are functions which depend upon the hydrodynamic and thermal boundary conditions imposed on the porous layer in the y-direction, and θ_c is the temperature profile for pure conduction regime.

Following Bejan [19], an equivalent energy flux condition in the x-direction can be imposed such that

$$C=\int_0^1\left(\frac{d\psi}{dy}\right)_x\theta\,dy \tag{17}$$

The substitution of Eqs. (15) and (16) into Eq. (17) gives the following equation

$$C=Ra^2C^3I_2+RaCI_1 \tag{18}$$

where the values of I_1 and I_2 are given respectively by

$$I_1=\int_0^1\frac{dF_b}{dy}\theta_c\,dy \tag{19}$$

$$I_2=\int_0^1\frac{dF_b}{dy}G_b\,dy \tag{20}$$

Dividing Eq. (18) by C, one obtains

$$C=\pm\frac{1}{Ra}\sqrt{\frac{(1-RaI_1)}{I_2}} \tag{21}$$

After evaluating I_1 and I_2, the axial temperature gradient C may be evaluated from Eq. (21) for given values of Ra, Da, K^* and θ. Physically, I_1 and I_2 are always positive and negative respectively. Considering the trivial solution ($C=0$) of Eqs. (18) and (21), one can retain that when $I_1<1/Ra$, two sets of solutions with positive and negative real roots of C exist, giving rise to convection cells in opposite directions. If $I_1>1/Ra$, $C=0$ is the only real root of (18) and there is no convection. Obviously, the marginal state determining the critical Rayleigh number Ra_c for the onset of convection is reached when $I_1=1/Ra$.

The Nusselt number Nu for the present problem is given by

$$Nu=\frac{\Delta T_c}{\Delta T} \tag{22}$$

where $\Delta T=T(0,0)-T(0,1)$ is the wall-to-wall dimensionless temperature difference and $\Delta T_c=1/2$ the corresponding value of pure conduction regime.

In the following parts of this section, we will consider situations when the bounding horizontal walls are both rigid or both stress-free, corresponding to several realistic cases studied in the past by many authors, on the basis of the isotropic condition of the porous medium.

3.1. Case 1: Both horizontal boundaries rigid

As allowed by the Brinkman-extended Darcy model, the no-slip boundary condition on the solid wall must be satisfied. That means a zero velocity on the rigid boundaries for any Darcy number Da. So, in this case, one has to assume the following boundary conditions

$$\text{At } y=0,\quad \psi=\frac{d\psi}{dy}=0,\quad \frac{d\theta}{dy}=-1 \tag{23}$$

$$\text{At } y=1,\quad \psi=\frac{d\psi}{dy}=0,\quad \frac{d\theta}{dy}=0 \tag{24}$$

The solutions to Eqs. (13) and (14) associated to boundary conditions (23) and (24) are given by Eqs. (15) and (16) where

$$F_b(y)=\frac{1}{2\varsigma b^{3/2}}\left\{\frac{\cosh\left[\varsigma\sqrt{b}(1/2-y)\right]}{\sinh\left(\varsigma\sqrt{b}/2\right)}-\coth\left(\varsigma\sqrt{b}/2\right)-\varsigma\sqrt{b}(y^2-y)\right\} \tag{25}$$

$$G_b(y) = -\frac{1}{2b}\left\{\frac{\sinh\left[\varsigma\sqrt{b}(1/2-y)\right]}{\varsigma^2 b \sinh\left(\varsigma\sqrt{b}/2\right)} + \frac{y}{\varsigma\sqrt{b}}\coth\left(\varsigma\sqrt{b}/2\right) + \frac{y^3}{3} - \frac{y^2}{2} - \frac{1}{\varsigma^2 b}\right\} \quad (26)$$

$$\theta_c = \frac{y^2}{2} - y \quad (27)$$

Substituting Eqs. (25) and (26) into Eqs. (19) and (20) and integrating, it is readily found that

$$I_1 = \frac{1}{24b}\left[1 + \frac{12}{\varsigma^2 b} - \frac{6}{\varsigma\sqrt{b}}\coth\left(\varsigma\sqrt{b}/2\right)\right] \quad (28)$$

and

$$I_2 = \frac{1}{4b^2}\left\{\frac{\sinh(\varsigma\sqrt{b}) - \varsigma\sqrt{b}}{2\varsigma^3 b^{3/2}\sinh^2\left(\varsigma\sqrt{b}/2\right)} - \frac{1}{30} + \frac{8}{\varsigma^4 b^2} + \frac{\coth\left(\varsigma\sqrt{b}/2\right)}{\varsigma^3 b^{3/2}}\left[\frac{\varsigma^2 b}{3} - 2 - \varsigma b\coth\left(\varsigma\sqrt{b}/2\right)\right]\right\} \quad (29)$$

Thus, it can be seen that the temperature gradient C may be evaluated from Eqs. (18), (21), (28) and (29) for given values of Ra, Da, K^* and θ. And then the stream function and temperature fields are known from Eqs. (11), (12) and (25)–(27).

Since the critical Rayleigh number Ra_c is related to the condition $I_1 = 1/Ra_c$, one can readily find from Eq. (28) that

$$Ra_c = 24b\left[1 + \frac{12}{\varsigma^2 b} - \frac{6}{\varsigma\sqrt{b}}\coth(\varsigma\sqrt{b}/2)\right]^{-1} \quad (30)$$

From Eq. (30) we can observe two limiting cases, the first one when ς is very large ($\varsigma \to \infty$), corresponding to the anisotropic porous medium for which $Ra_c = 24b$ and the second one $\varsigma \to 0$ (i.e., viscous fluid case where anisotropic effects are irrelevant) for which $Ra_c^* = 1440$ (where $Ra_c^* = Ra/Da$ is the Rayleigh number for viscous fluid). These results are in agreement with those found by Vasseur and Robillard [8] when the porous medium is isotropic (i.e., $K^* = 1 (b = 1)$) and by Kulacki and Goldstein [20] who have considered the same problem in the case of pure fluid situation for which the onset of convection occurs at zero wave number when $Ra_c^* = 1433.6$, using linear and energy stability theory.

From the temperature distribution, the Nusselt number is given by Eq. (22) as

$$Nu = \left[1 - \frac{RaC^2}{b}\left(\frac{1}{6} + \frac{2}{\varsigma^2 b} - \frac{1}{\varsigma\sqrt{b}}\coth\left(\varsigma\sqrt{b}/2\right)\right)\right]^{-1} \quad (31)$$

3.2. Case 2: Both horizontal boundaries stress-free

In this situation, the boundary conditions applied to the physical model are

$$\text{At } y = 0, \quad \psi = \frac{d^2\psi}{dy^2} = 0, \quad \frac{d\theta}{dy} = -1 \quad (32)$$

$$\text{At } y = 1, \quad \psi = \frac{d^2\psi}{dy^2} = 0, \quad \frac{d\theta}{dy} = 0 \quad (33)$$

One can notice that the velocity gradient is zero at the boundary for any value of Da and of anisotropic parameters of the porous matrix. So, this hydrodynamic condition is valid for the Brinkman model only. Due to the no-slip condition which has to be satisfied at the solid wall, that constraint does not exist according to the Darcy model. Consequently, the functions F_b and G_b satisfying the boundary conditions (32) and (33) are given by

$$F_b(y) = \frac{1}{(\varsigma b)^2}\left\{\frac{\cosh\left[\varsigma\sqrt{b}(1/2-y)\right]}{\cosh\left(\varsigma\sqrt{b}/2\right)} - \frac{\varsigma^2 b}{2}(y^2 - y) - 1\right\} \quad (34)$$

$$G_b(y) = -\frac{1}{b}\left\{\frac{\sinh\left[\varsigma\sqrt{b}(1/2-y)\right]}{\varsigma^3 b^{3/2}\cosh\left(\varsigma\sqrt{b}/2\right)} + \frac{y}{\varsigma^2 b} + \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^2}{2}\right) - \frac{\tanh\left(\varsigma\sqrt{b}/2\right)}{\varsigma^3 b^{3/2}}\right\} \quad (35)$$

Substituting Eqs. (34), (35) and (27) into Eqs. (19) and (20) and integrating yield

$$I_1 = \frac{1}{24b}\left[1 - \frac{12}{\varsigma^2 b} + \frac{24}{\varsigma^3 b^{3/2}}\tanh\left(\varsigma\sqrt{b}/2\right)\right] \quad (36)$$

and

$$I_2 = \frac{\sinh(\varsigma\sqrt{b}) - \varsigma b^3}{2\varsigma^5 b^{9/2}\cosh^2(\varsigma\sqrt{b}/2)} - \frac{1}{120b^2} + \frac{1}{6\varsigma^2 b^3} - \frac{3}{\varsigma^4 b^4} + \frac{6}{\varsigma^5 b^{9/2}}\tanh(\varsigma\sqrt{b}/2) \quad (37)$$

Considering the previous results, the temperature gradient C may be calculated by substituting Eqs. (36) and (37) into Eq. (18) or (21). Thus, the stream function and temperature fields can be deduced from Eqs. (11), (12), (34), (35) and (27).

The critical Rayleigh number for the onset of convection is obtained from the relation characterizing the marginal state ($I_1 = 1/Ra$), in which I_1 is replaced by its expression (36) to give

$$Ra_c = 24b\left[1 - \frac{12}{\varsigma^2 b} + \frac{24}{\varsigma^3 b^{3/2}}\tanh\left(\varsigma\sqrt{b}/2\right)\right]^{-1} \quad (38)$$

such that, for the Darcy case corresponding to the anisotropic porous media ($\varsigma \to \infty$), $Ra_c = 24b$ and for the viscous fluid case when ς is very small ($\varsigma \to 0$), $Ra_c^* = 240$. It is immediately seen that for the case of an isotropic porous medium ($K^* = 1$), the result found here is in agreement with the solution predicted by Vasseur and Robillard [8].

From the temperature distribution and using Eq. (35), the Nusselt number given by Eq. (22) becomes

$$Nu = \left\{ 1 - \frac{RaC^2}{b}\left[\frac{4\tanh(\varsigma\sqrt{b}/2)}{\varsigma^3 b^{3/2}} - \frac{2}{\varsigma^2 b} + \frac{1}{6}\right]\right\}^{-1} \tag{39}$$

4. Application to the convection due to a sudden heating or cooling

As the parallel flow analysis provides a smooth transition between the pure fluid layer and the Darcy porous medium, in the establishment of the critical Rayleigh for the onset of convection (see for example Vasseur and Robillard [8]), a step-function temperature profile and a piecewise-linear profile will be applied in this section to investigate a closed-form solution of Ra_c as a function of Da and of anisotropic parameters for all the cases considered previously.

4.1. Step-function temperature profile

Following Rudraiah et al. [7] and other authors, the step-function temperature profile is defining as the basic temperature which drops suddenly by an amount ΔT at $y = \eta$ such that

$$\theta_c = \begin{cases} 0, & 0 \leqslant y < \eta \\ -1, & \eta < y \leqslant 1 \end{cases} \tag{40}$$

The governing equations are still given by Eqs. (9) and (10) with the uniform sink term S set to zero. Using Eqs. (19), (25) and (40) and the marginal state condition for the onset of convection, the critical Rayleigh number when both boundaries are rigid is obtained as

$$Ra_c = 2b\left\{\eta(1-\eta) + \frac{1}{\varsigma b^{1/2}}\left[\frac{\cosh\left[\varsigma\sqrt{b}(1/2-\eta)\right]}{\sinh\left(\varsigma\sqrt{b}/2\right)} - \coth\left(\varsigma\sqrt{b}/2\right)\right]\right\}^{-1} \tag{41}$$

It is seen from Eq. (41) that Ra_c decreases from ∞ to a minimum value and then increases again to ∞ as η from 0 to 1. The minimum value of Ra_c reached at $\eta = 1/2$ (corresponding to a midway between the boundaries) is known by

$$Ra_c = \left[\frac{1}{8b} + \frac{1-\cosh(\varsigma\sqrt{b}/2)}{2\varsigma b^{3/2}\sinh(\varsigma\sqrt{b}/2)}\right]^{-1} \tag{42}$$

such that one gets $Ra_c = 8b$ when $\varsigma \to \infty$ and $Ra_c^* = Ra_c\varsigma^2 = 384$. We found that for the particular case when $b = 1$ (corresponding to the isotropic porous medium), the result is in agreement with that obtained Vasseur and Robillard [8].

Considering the case when both boundaries are stress-free, the critical Rayleigh number obtained from Eqs. (19), (34) and (40) and the marginal state condition for the onset convection, is given by

$$Ra_c = \left\{\frac{\eta(1-\eta)}{2b} - \frac{1}{\varsigma^2 b^2}\left[1 - \frac{\cosh\left[\varsigma\sqrt{b}(1/2-\eta)\right]}{\cosh(\varsigma\sqrt{b}/2)}\right]\right\}^{-1} \tag{43}$$

The minimum value of Ra_c corresponding to $\eta = 0.5$ in this situation is then

$$Ra_c = \left\{\frac{1}{8b} - \frac{1}{\varsigma^2 b^2}\left[1 - \frac{1}{\cosh(\varsigma\sqrt{b}/2)}\right]\right\}^{-1} \tag{44}$$

such that, when $\varsigma \to 0$ (for a viscous fluid layer) $Ra_c^* = 76.8$ which is the known value (see Nield [5]).

4.2. Piecewise-linear profile

This kind of profile considered here to approximate the temperature for heating from below is defined as

$$\theta_c = \begin{cases} -y/\eta, & 0 \leqslant y < \eta \\ -1, & \eta < y \leqslant 1 \end{cases} \tag{45}$$

From Eqs. (19), (25) and (45), and the marginal state condition for the onset of convection, the critical Rayleigh number obtained when both boundaries are rigid is then

$$Ra_c = \left\{\frac{(3\eta - 2\eta^2)}{12b} + \frac{1}{2\eta\varsigma^2 b^2}\left[1 - \frac{\sinh\left[\varsigma\sqrt{b}(1/2-\eta)\right]}{\sinh(\varsigma\sqrt{b}/2)}\right] - \frac{\coth(\varsigma\sqrt{b}/2)}{2\varsigma b^{3/2}}\right\}^{-1} \tag{46}$$

which has a minimum of $96b/9$, attained at $\eta = 0.75$ when $\varsigma \to \infty$ and a minimum of 601.1 at $\eta = 0.724$, when $\varsigma \to 0$.

For the case when the porous layer is heated from below and cooled from the above by a constant heat flux, the critical Rayleigh number is obtained, by substituting $\eta = 1$ into (46), as

$$Ra_c = \left[\frac{1}{12b} + \frac{1}{\varsigma^2 b^2} - \frac{\coth(\varsigma\sqrt{b}/2)}{2\varsigma b^{3/2}}\right]^{-1} \tag{47}$$

such that, when $\varsigma \to \infty$ (for an anisotropic porous medium) $Ra_c = 12b$ and $Ra_c^* = 720$ when $\varsigma \to 0$ (for a pure fluid medium).

When both boundaries are stress-free, it is readily found that

$$Ra_c = \left\{ \frac{(3\eta - 2\eta^2)}{12b} - \frac{1}{\varsigma^2 b^2} + \frac{1}{\eta \varsigma^3 b^{5/2}} \left[\tanh(\varsigma\sqrt{b}/2) - \frac{\sinh\left[\varsigma\sqrt{b}(1/2-\eta)\right]}{\cosh(\varsigma\sqrt{b}/2)} \right] \right\}^{-1} \quad (48)$$

yielding a minimum value of $Ra_c = 96b/9$, attained at $\eta = 0.75$ when $\varsigma \to \infty$ and a minimum value of $Ra_c^* = 105.57$ at $\eta = 0.744$, when $\varsigma \to 0$.

When $\eta = 1$ (for heating from below and cooling from the above), Eq. (48) reduces to

$$Ra_c = \left[\frac{1}{12b} - \frac{1}{\varsigma^2 b^2} + \frac{2}{\varsigma^3 b^{5/2}} \tanh(\varsigma\sqrt{b}/2) \right]^{-1} \quad (49)$$

such that $Ra_c = 12b$ when $\varsigma \to \infty$ (for anisotropic porous media) and $Ra_c^* = 120$ when $\varsigma \to 0$ (for a pure fluid layer heated from below by a constant heat flux).

5. Results and discussion

The minimum value of the critical Rayleigh number Ra_c (corresponding to the midway between the boundaries ($\eta = 1/2$)) has been plotted as a function of Da for the step-function temperature profile when both the boundaries are rigid, the inclination of the principal axes is maintained at $\theta = 0°$ and for various values of K^*. The results indicate that, in comparison with the isotropic situation ($K^* = 1$), the critical Rayleigh number for the onset of convection, Ra_c, is enhanced when $K^* > 1$ ($K^* = 2.5$) and reduced when $K^* < 1$ ($K^* = 0.25$). This trend follows from the fact that Ra_c (and Ra) depending upon K_1, are proportional to the latter and an increase (decrease) in K^* ($= K_1/K_2$) corresponds to an increase (decrease) in the permeability K_1. On the other hand, when the principal axes of anisotropy are aligned with the gravity vector, it is observed that the characteristic parameter for the appearance of the convection increases (decreases) when the permeability in the vertical direction is greater (lower) than that in the horizontal direction. Fig. 1 also indicates that the critical Rayleigh number predicted by Darcy's model ($Ra_c = 8b$), illustrated by dashed lines, starts to deviate from Brinkman's model at a Darcy number that increases as K^* is made larger. When the Darcy number is large enough, the results show that the curves, for a given value of K^*, tend asymptotically towards the pure fluid situation for $Ra_c^* = 384$ (represented in a dotted line). The Darcy number required to reach this limit increases as the value of K^* is made larger.

The effect of varying Da on the onset of convection corresponding to a step-function temperature profile in the case when the horizontal boundaries are rigid, is depicted in Fig. 2 where the critical Rayleigh number, given by Eq. (42) is plotted as a function of Da for $K^* = 0.25$ and various values of γ. It is clear from Fig. 2 that, when Da is made weaker, Ra_c tends asymptotically towards a constant value that depends on γ. The limit ($Da \to 0$) represented by dashed lines, corresponds as notified previously, to a pure Darcy medium situation ($Ra_c = 8b$). Thus, $Ra_c = 2$ when $\gamma = 0°$ and $Ra_c = 8$ when $\gamma = 90°$. It is noticed from Fig. 2 that, when $K^* < 1$, the critical Rayleigh number is enhanced as the anisotropy orientation is increased from zero towards 90°. This behaviour can be easily demonstrated by the fact, when $K^* < 1$ (i.e., $K_1 < K_2$), the critical Rayleigh

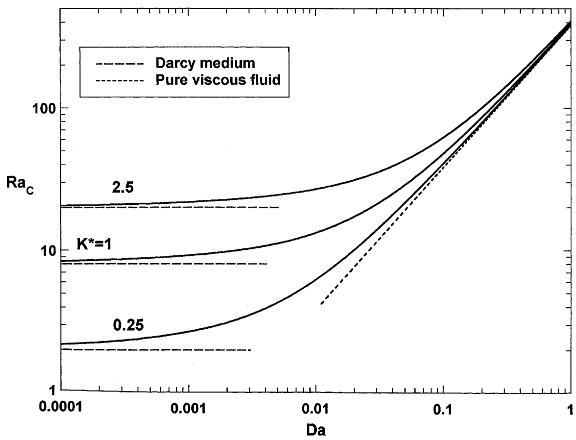

Fig. 1. Effect of the Darcy number Da on Ra_c corresponding to a step-function temperature profile in the case when both horizontal boundaries are rigid, for $\gamma = 0°$ and various values of K^*.

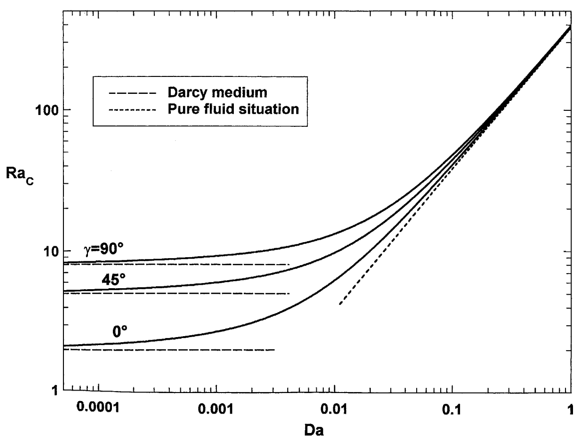

Fig. 2. Effect of the Darcy number Da on Ra_c corresponding to a step-function temperature profile in the case when both horizontal boundaries are rigid, for $K^* = 0.25$ and various values of γ.

number proportional to $b^{3/2}$, increases, as b increases from 0.25 (for $\gamma = 0°$) to 1 (for $\gamma = 90°$). Inversely, when ($K^* > 1$), the results (not presented here) show that Ra_c decreases with an increase in γ for the same reason explained earlier. As Da increased, the boundary frictional resistance becomes gradually more important and adds to the bulk frictional drag induced by the solid matrix to slow up the convection until the neighbourhood of the pure fluid situation represented in a dotted line. As a result, the effect of varying the anisotropy of the porous medium becomes less and less important. When Da is high enough, i.e., when the resistance resulting from the boundary effects is predominant with respect to that due to the solid matrix, the present solution approaches that for a pure viscous fluid, independently of the anisotropy of the porous medium. This situation is reached when $Da \simeq 0.2$ for the conditions of Fig. 2.

The influence of the anisotropy orientation, γ on the critical Rayleigh number corresponding to a piecewise-linear profile in the situation when both the horizontal boundaries are stress-free, is presented in Fig. 3 for $Da = 10^{-3}$ and various values of K^*. In an isotropic porous medium ($K^* = 1$), the critical Rayleigh number, Eq. (49), is independent of γ. Indeed, the result ($Ra_c = 12$) obtained in this limit is in agreement with that found by Vasseur and Robillard [8]. In general, a symmetry of the results with respect to $\gamma = 90°$ is observed in Fig. 3. This follows from the fact that it can be easily shown that the expression of b, Eq. (8), being a function of K^* and γ, gives the same value with K^* and $(\pi - \gamma)$, such that we can limit the discussion to $0 \leqslant \gamma \leqslant 90°$. Fig. 3 also indicates that for $K^* < 1$, Ra_c is minimum at $\gamma = 0°$ for which the permeability in the vertical direction is minimum and maximum at $\gamma = 90°$ for which the permeability in the vertical direction is maximum. For this situation, the minimum of curves at $\gamma = 0°$ varies from zero when $K^* = 10^{-3}$ towards 12 when $K^* = 1$. Furthermore, the maximum of these curves at 90° is sole and corresponds to the critical Rayleigh number characterizing the isotropic case ($Ra_c = 12$). Inversely, when $K^* > 1$, the critical Rayleigh number is now maximum at $\gamma = 0°$ and minimum at $\gamma = 90°$. For this case, the maximum of Ra_c at $\gamma = 0°$ increases as K^* is made larger and the minimum observed for all curves at $\gamma = 90°$ corresponds to that obtained for an isotropic porous medium. The behaviour of Ra_c can be deduced from the first and second derivatives of Ra_c (Eq. (49)) with respect to γ. Thus, when $K^* > 1$ ($K^* < 1$), Ra_c is maximum (minimum) at $\gamma = 0°$ and minimum (maximum) at $\gamma = 90°$. It follows from these results that a maximum (minimum) critical Rayleigh number is reached when the orientation of the principal axis with higher permeability of the anisotropic porous medium is parallel (perpendicular) to the gravity. Similar results have been reported in the past [21,22] concerning the convective heat transfer in a vertical porous cavity heated from the sides.

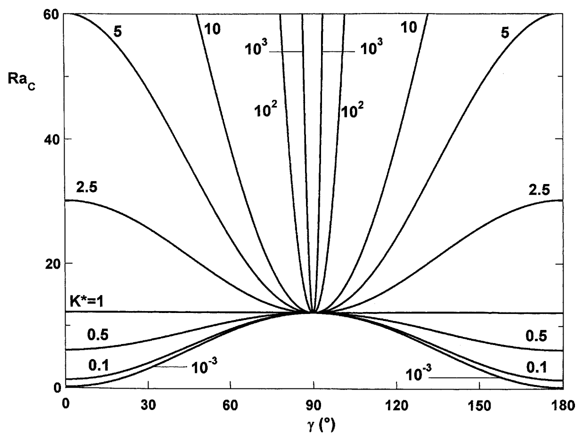

Fig. 3. Effect of the anisotropy orientation angle γ on Ra_c corresponding to a piecewise-linear profile in the case when both horizontal boundaries are stress-free, for $Da = 10^{-3}$ and various values of K^*.

6. Conclusions

On the basis of the generalized Brinkman-extended Darcy model and using the parallel flow analysis, closed-form solutions for the critical Rayleigh numbers and the Nusselt numbers are obtained to investigate the influence of hydrodynamic anisotropy on natural convection in a horizontal shallow porous layer heated from below by a constant heat flux, while the other boundaries are maintained adiabatic. Applying the results to solve problem of convection arising either from sudden heating or cooling in a fluid-saturated porous medium, the following remarks are in order

(1) Both the permeability ratio K^* and the inclination angle γ of the principal axes have a strong influence on the critical Rayleigh number for the onset of convection.
(2) The critical Rayleigh number increases (decreases) when the permeability in the vertical direction is greater (lower) than that in the horizontal direction, when $\gamma = 0°$.
(3) Upon increasing the anisotropy orientation from 0° towards 90°, the critical Rayleigh number is enhanced when $K^* < 1$ and is weakened when $K^* > 1$.
(4) A maximum (minimum) of the critical Rayleigh number is reached when the orientation of the principal axis with higher permeability of the anisotropic porous medium is parallel (perpendicular) to the gravity.

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